Algebraic Geometry Notes

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Introduction

Rings are always commutative with unity.

Chapter 1

Affine Space

1.1 Affine Varieties and the Nullstellensatz

In algebraic geometry we are interested in studying spaces which are locally described by polynomial equations.

Definition 1.1

Let $F \in \mathbb{R}[x, y]$. Then the **vanishing set** or zero locus of F is

$$V(F) = \left\{(x, y) \in \mathbb{R}^2 : F(x, y) = 0\right\}$$

This extends the construction of many familiar spaces, such as the circle:

$$S^1 = V(f), \quad f(x, y) = x^2 + y^2 - 1$$

These equations can give information about the spaces themselves. For instance, we say that a point in the vanishing set is *singular* if $\nabla F = 0$ at that point. For S^1, there are no singular points, while for the more complicated space

$$y^2 - x^2(x + 1) = 0$$

there is a singular point at $(0, 0)$, which is seen to be a crossing point of this curve. Thus, we intuitively see that singular points should correspond somehow to a failure to be smooth or a manifold.

This corresponds to the general situation in algebraic geometry, where we are able to apply tools from commutative algebra to analyze our spaces, as well as giving geometric context and intuition for problems in algebra.

It is worth noting that the solution sets to polynomial equations depend on the space we work over. For instance, $x^2 + y^2 - 1 = 0$ is satisfied at four discrete points over $\mathbb{Z}^2$, a circle over $\mathbb{R}^2$, and over $\mathbb{C}^2$ the solution set is basically a sphere minus one point. One could also work over finite fields, and so on. For now we will be working over algebraically closed fields, in particular $\mathbb{C}$.

Definition 1.2

Let K be an algebraically closed field. Then we define the **affine n-space** over K to be

$$\mathbb{A}_K^n = \{(a_1, \ldots, a_n) : a_i \in K\}$$

We may also denote affine n-space by $\mathbb{A}^n$ when K is obvious.

Definition 1.3

Let $S \subseteq K[x_1, \ldots, x_n]$ be a collection of polynomials. Then their **vanishing locus** is

$$V(S) = \{x \in \mathbb{A}^n : f(x) = 0 \text{ for all } f \in S\} \subseteq \mathbb{A}^n$$

Any set of this form is called an **affine variety**.

Proposition 1.1

If $S_1 \subseteq S_2 \subseteq K[x_1, \ldots, x_n]$ then $V(S_1) \supseteq V(S_2)$.

Proposition 1.2

If f, g are polynomials then

$$V(f, g) = V(f) \cap V(g)$$

and

$$V(fg) = V(f) \cup V(g)$$

More generally, for collections $S_1, S_2 \subseteq K[x_1, \ldots, x_n]$,

$$V(S_1 \cup S_2) = V(S_1) \cap V(S_2)$$

and

$$V(S_1 S_2) = V(S_1) \cup V(S_2)$$

where $S_1 S_2$ is the pointwise product

$$S_1 S_2 = \{fg : f \in S_1, g \in S_2\}$$

Even more generally, we can take arbitrary unions of collections $\bigcup_{\lambda \in \Lambda} S_\lambda$ (equivalently, arbitrary intersections of the varieties $\bigcap_{\lambda \in \Lambda} V(S_\lambda)$), but only *finite* products (equivalently, unions).

We note that the pointwise product result requires the fact that K is a domain (since we need the fact that $fg = 0 \implies f = 0$ or $g = 0$).

Proposition 1.3

If $X \subseteq \mathbb{A}^n, Y \subseteq \mathbb{A}^m$ are affine varieties, then $X \times Y \subseteq \mathbb{A}^{n+m}$ is also an affine variety.

Proof. Let $S_X \in K[x_1, \ldots, x_n]$ and $S_Y \in K[y_1, \ldots, y_m]$ be such that $X = V(S_X), Y = V(S_Y)$. Then project these naturally into $K[x_1, \ldots, x_{n+m}]$ by converting each formal variable y_i to x_{n+i}, and it follows that $X \times Y = V(S'_X, S'_Y)$. □

Example 1.1

In $\mathbb{A}^1$, the only affine varieties are $\mathbb{A}^1$, and finite sets. This can be seen by noting that we immediately have

$$V(S) \subseteq V(f)$$

whenever $f \in S$. Then, either S only contains the zero polynomial, or else f can be chosen to be nonzero, hence to have finitely many roots.

This logic also shows that any variety in $\mathbb{A}^1$ can be described by a single polynomial equation $(z - z_1) \cdots (z - z_n) = 0$, or the zero polynomial. (For general S this polynomial need not actually be in S, but if S is an ideal then it is.) We would like to have a similar finiteness result for varieties in $\mathbb{A}^n$. The following will demonstrate this.

Definition 1.4

Recall that if $S \subseteq K[x_1, \ldots, x_n]$, then the generated **ideal** is

$$\langle s \rangle = \{r_1 f_1 + \ldots + r_m f_m : r_i \in K[x_1, \ldots, x_n], f_i \in S\}$$

Proposition 1.4

$V(S) = V(\langle S \rangle)$.

Proof. To show that $V(\langle S \rangle) \subseteq V(S)$, we just note that $S \subseteq \langle S \rangle$.

To see that $V(S) \subseteq V(\langle S \rangle)$, at any element $x \in V(S)$, each $f \in S$ vanishes, so any polynomial in $\langle S \rangle$ vanishes as well. □

Theorem 1.5: Hilbert's Basis Theorem

If a ring R is Noetherian then $R[x]$ is as well.

Proof. See Section A.1. □

In particular note that fields are always Noetherian, so we see that $\langle S \rangle$ itself is finitely generated.

Corollary 1.6

Any affine variety can be written as $X = V(f_1, \dots, f_m) \subseteq \mathbb{A}^m$.

Thus we see that the theory of varieties can be written entirely in terms of ideals, which allows us to enjoy finiteness properties.

Definition 1.5

If J is an ideal in R then the **radical** of J is

$$\sqrt{J} = \{x \in R : \exists m \text{ s.t. } x^m \in J\}$$

Lemma 1.7

For I, J two ideals, $\sqrt{IJ} = \sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$.

Proof. First, we have $IJ \subseteq I \cap J$, since $SI \subseteq I$ for any subset $S \subseteq R$ and any ideal I. Therefore $\sqrt{IJ} \subseteq \sqrt{I \cap J}$.

If $x \in \sqrt{I \cap J}$ with $x^m = z$, $z \in I \cap J$, then plainly $x \in \sqrt{I}, \sqrt{J}$ as well so $\sqrt{I \cap J} \subseteq \sqrt{I} \cap \sqrt{J}$.

Lastly, if $x \in \sqrt{I} \cap \sqrt{J}$, then $x^m = y \in I$ and $x^n = z \in J$ for some y, j. It follows that $x^{m+n} = yz \in IJ$. So $\sqrt{IJ} \subseteq \sqrt{I} \cap \sqrt{J}$, completing the circular inclusion. □

Lemma 1.8

If J, J_1, J_2 are ideals in $K[x_1, \dots, x_n]$, then

(a) $V(\sqrt{J}) = V(J)$.

(b) $V(J_1) \cup V(J_2) = V(J_1 J_2) = V(J_1 \cap J_2)$.

(c) $V(J_1) \cap V(J_2) = V(J_1 + J_2)$.

Proof. (a) $V(\sqrt{J}) \subseteq V(J)$ because $J \subseteq \sqrt{J}$. In the other direction, if $x \in V(J)$, then for any $g \in V(\sqrt{J})$, $g^m(x) = 0$ for some m. Since K is a field, $g(x) = 0$ so $x \in V(\sqrt{J})$.

(b) The first equality is a special case of the version for subsets. The second follows from the lemma $\sqrt{IJ} = \sqrt{I \cap J}$.

(c) This is the previous version, except that we note that $J_1 + J_2 = \langle J_1 \cup J_2 \rangle$.

□

At this point, we have constructed a mapping that takes ideals to affine varieties. To better understand this correspondece, we now construct the inverse map.

Definition 1.6

Let $X \subseteq \mathbb{A}^n$ be a subset. Then **vanishing ideal** of X is

$$I(X) = \{f \in K[x_1, \ldots, x_n] : f|_X = 0\}$$

For similar reasons as before, this is indeed an ideal. In particular it is a radical ideal. Also, the map $X \mapsto I(X)$ is inclusion-reversing, so if $X_1 \subseteq X_2$ then $I(X_1) \supseteq I(X_2)$.

While $X \mapsto I(X)$ and $J \mapsto V(J)$ are maps from sets to ideals and vice versa, we need to clarify the manner in which they compose, since they are not strictly inverses of each other.

Theorem 1.9: Nullstellensatz

(a) If $J \subseteq K[x_1, \ldots, x_n]$ is an ideal then $I(V(J)) = \sqrt{J}$.

(b) If $X \subseteq \mathbb{A}^n$ is an affine variety then $V(I(X)) = X$.

Proof. (a) It is clear that any f with $f^m \in J$ must vanish on $V(J)$ by our previous work, so $\sqrt{J} \subseteq V(J)$. The other direction is the actually difficult part of this proof and is not demonstrated. It is equivalent to the statement that if f vanishes on $V(J)$ then $f^m \in J$ for some m.

(b) Suppose $X = V(J)$. Then $J \subseteq \sqrt{J} = I(X)$ (by part a) so $V(I(X)) \subseteq X$. In the other direction, $I(X)$ is defined to be functions which vanish on X, so necessarily X lies in their vanishing locus. □

This shows that if we restrict the previous mapping between ideals and varieties to only the radical ideals, then we get a bijection. This bijection is inclusion-reversing.

Corollary 1.10

There is a bijective correspondence between maximal ideals and points $a = (a_1, \ldots, a_n) \in \mathbb{A}^n$.

Proof. Maximal ideals are maximal among radical ideals (since the radical of a proper ideal is proper). By the inclusion reversing bijection, they correspond to *minimal* nonempty varieties, which are singular points. □

In particular the bijection can be written as $a \mapsto \langle x_1 - a_1, \ldots, x_n - a_n \rangle$.

Lemma 1.11

Let $X_1, X_2 \subseteq \mathbb{A}^n$ be affine varieties. Then

(a) $I(X_1 \cup X_2) = I(X_1) \cap I(X_2)$,

(b) $I(X_1 \cap X_2) = \sqrt{I(X_1) + I(X_2)}$.

Proof. (a) By the inclusion reversing bijection,

$$\begin{cases} X_1 \subseteq X_1 \cup X_2 \\ X_2 \subseteq X_1 \cup X_2 \end{cases} \implies I(X_1 \cup X_2) \subseteq I(X_1) \cap I(X_2)$$

and any function which vanishes on X_1, X_2 vanishes on their union, so $I(X_1) \cap I(X_2) \subseteq I(X_1 \cup X_2)$.

(b) We have

$$I(X_1 \cap X_2) = I(V(I(X_1)) \cap V(I(X_2))) = I(V(I(X_1) + I(X_2))) = \sqrt{I(X_1) + I(X_2)} \quad \square$$

Example 1.2

Writing $z = x + iy$ in $\mathbb{A}^1_{\mathbb{C}}$, let X_1, X_2 be the subsets $\{x = 0\}, \{y = 0\}$, respectively. Note that as before these are *not* algebraic varieties, so the above Lemma does not hold. In fact it fails, since the vanishing ideal of both is the zero ideal, but their intersection is the origin, so:

$$I(X_1 \cap X_2) = \langle x \rangle \neq 0 = \sqrt{I(X_1) + I(X_2)}$$

Corollary 1.12: Weak Nullstellensatz

If $J \subseteq K[x_1, \ldots, x_n]$ is an ideal, then the following are equivalent:

(a) $V(J) = \varnothing$,

(b) $J = K[x_1, \ldots, x_n]$,

(c) $1 \in J$.

Proof. (b) $\iff$ (c) is basic ring theory. Now note that if $1 \in J$ then clearly $V(J) = \varnothing$, and if $V(J) = \varnothing$ then 1 vanishes on $V(J)$, so $1 \in \sqrt{V(J)} \implies 1 \in J$. $\square$

1.2 Subvarieties

We now study varieties which live inside of other varieties, instead of simply $\mathbb{A}^n$.

Definition 1.7

Let $X \subseteq \mathbb{A}^n$ be an affine variety. Then the **coordinate ring** of X is

$$A(X) = \{f|_X : f \in K[x_1, \ldots, x_n]\} \subseteq K^{(K^n)}$$

Note that the elements of $A(X)$ are considered to be functions represented by polynomials, not polynomials, and only need to be equal as functions. For instance, $x^2 + x$ is the zero function on $\mathbb{F}_2$, but is not equal as a polynomial to the zero polynomial. ($\mathbb{F}_2$ is not algebraically closed, but this is just for illustration.)

Lemma 1.13

$A(X)$ is a K-algebra.

Recall that a K-algebra can equivalently be defined either as a K-vector space with a bilinear product, or as a ring A with a ring homomorphism from K into $Z(A)$.

Proof. $A(X)$ is clearly a ring, and the homomorphism takes $\lambda \in K$ to the constant polynomial $f \equiv \lambda$. □

Example 1.3

If $X \subseteq \mathbb{A}^1$ is a finite set with r points, then by Lagrange interpolation we can define a polynomial with any prescribed values on these points. So $A(X) = K^r$.

Proposition 1.14

Let $X \subseteq \mathbb{A}^n$ be an affine variety. Then $f \mapsto f|_X$ is a surjective ring homomorphism with kernel $I(X)$, so that

$$A(X) = K[x_1, \ldots, x_n] / I(X)$$

Proof. Surjectivity is by definition. Then, we have injectivity since $f|_X = 0$ if and only if f vanishes on X, which is to say that $f \in I(X)$. □

In particular if $X = V(J)$ then

$$A(V(J)) = K[x_1, \ldots, x_n] / \sqrt{J}$$

Corollary 1.15

$A(\mathbb{A}^n) = K[x_1, \ldots, x_n]$.

Proof. We have $\mathbb{A}^n = V(\langle 0 \rangle)$, so $I(\mathbb{A}^n) = I(V(\langle 0 \rangle)) = \sqrt{\langle 0 \rangle} = \langle 0 \rangle$. Hence

$$A(\mathbb{A}^n) = K[x_1, \ldots, x_n]$$

□

This shows that the theory of polynomials on $\mathbb{A}^n$ is really the theory of polynomial *functions* on $\mathbb{A}^n$, so there is no generality of studying polynomial functions on varieties. In this way, we can see that our previous geometric correspondence from ideals in $A(\mathbb{A}^n)$ should also give some meaning to ideals in general $A(X)$. These turn out to be the subvarieties.

Definition 1.8

Fix an affine variety $Y \subseteq \mathbb{A}^n$. Then for any $S \subseteq A(Y)$, define its **vanishing locus** to be

$$V(S) = V_Y(S) = \{x \in Y : f(x) = 0 \forall f \in S\}$$

A subset which is the vanishing locus of some $S \subseteq A(Y)$ is called an **affine subvariety**.

Correspondingly, for any subset $X \subseteq Y$, we define the **vanishing ideal** of X in Y to be

$$I(X) = I_Y(X) = \{f \in A(Y) : f|_X \equiv 0\}$$

This new correspondence behaves almost exactly the same as the one for varieties in $\mathbb{A}^n$.

Proposition 1.16

Let $Y \subseteq \mathbb{A}^n$ be an affine variety.

(a) $I_Y(X)$ is a (radical) ideal,

(b) any subvariety $X \subseteq Y$ is a variety in $\mathbb{A}^n$,

(c) any variety $X \subseteq \mathbb{A}^n$ such that $X \subseteq Y$ is also an affine subvariety of Y,

(d) whenever $X \subseteq Y$ is a subvariety,

$$A(X) \cong {}^{A(Y)}\!/_{I(X)}$$

(e) I_Y, V_Y give an inclusion reversing, bijective correspondence between affine subvarieties of Y and radical ideals of $A(Y)$,

(f) $V(J_1) \cup V(J_2) = V(J_1 J_2) = V(J_1 \cap J_2)$.

(g) $V(J_1) \cap V(J_2) = V(J_1 + J_2)$.

(h) $I(X_1 \cup X_2) = I(X_1) \cap I(X_2)$,

(i) $I(X_1 \cap X_2) = \sqrt{I(X_1) + I(X_2)}$.

Proof. (a) Clear by lifting to polynomials.

(b) Let $X = V_Y(J)$ for an ideal $J \subseteq A(Y)$. Let π be the quotient map from $K[x_1, \ldots, x_n]$ by $I(Y)$, and let $I = \pi^{-1}(J)$. Then since $X \subseteq Y$, for $x \in X$, $\pi g(x) = 0$ if and only if $g(x) = 0$. So

$$\begin{aligned} X = V_Y(J) = \{x : g'(x) = 0 \forall g' \in J\} &= \left\{x : \pi g(x) = 0 \forall g \in \pi^{-1}(J)\right\} \\ &= \left\{x : g(x) = 0 \forall g \in \pi^{-1}(J)\right\} = V(\pi^{-1}(J)) \end{aligned}$$

and X is a variety.

(c) Let $X = V(J)$. Then by the same logic as above,

$$\begin{aligned} X = V(J) = \{x : g(x) = 0 \forall g \in J\} &= \{x : \pi g(x) = 0 \forall g \in J\} \\ = \{x : g'(x) = 0 \forall g' \in \pi(J)\} &= V_Y(\pi(J)) \end{aligned}$$

so X is an affine subvariety of Y.

(d)

□

If we are given two varieties $X \subseteq \mathbb{A}^n, Y \subseteq \mathbb{A}^m$ which are cut out by

$$I(X) \subseteq K[x_1, \dots, x_n], \quad I(Y) \subseteq K[y_1, \dots, y_m]$$

then we know the product is a variety $\mathbb{A}^{n+m}$, which is cut out by taking

$$I_{X\times Y} = I(X)K[x_1, \dots, x_n, y_1, \dots, y_m] + I(Y)K[x_1, \dots, x_n, y_1, \dots, y_m]$$

This is a reflection of a more general fact concerning tensor products, since

$$K[x_1, \dots, x_n, y_1, \dots, y_m] = K[x_1, \dots, x_n] \otimes_K K[y_1, \dots, y_m]$$

Proposition 1.17

Let $I_A \subseteq A, I_B \subseteq B$ be ideals, and let C be a ring with morphisms into A, B. Then

$$I_{A,B} = I_A(A \otimes_C B) + I_B(A \otimes_C B)$$

is an ideal of $A \otimes_C B$, and the map

$$\begin{aligned} \left(A/I_A \right) \otimes_C \left(B/I_B \right) &\to A \otimes_C B / I_{A,B} \\ (a + I_A) \otimes_C (b + I_B) &\mapsto a \otimes_C b + I_{A,B} \end{aligned}$$

is an isomorphism.

Proposition 1.18

Let $X \subseteq \mathbb{A}^n, Y \subseteq \mathbb{A}^m$ with $I_{X\times Y}$ defined as above. Then $I(X \times Y) = I_{X\times Y}$ and

$$A(X \times Y) = A(X) \otimes_K A(Y)$$

Proof. It is clear that $V(I_{X\times Y}) = X \times Y$, so by the Nullstellensatz,

$$I(X \times Y) = \sqrt{I_{X\times Y}}$$

So we need to show $I_{X\times Y}$ is radical, which is equivalent to saying that

$$K[x_1, \dots, x_n, y_1, \dots, y_m] / I_{X\times Y}$$

is reduced. By the previous proposition we have that

$$K[x_1, \dots, x_n, y_1, \dots, y_m] \big/ I_{X\times Y}$$
$$\cong \left(K[x_1, \dots, x_n] \big/ I(X) \right) \otimes_K \left(K[y_1, \dots, y_m] \big/ I(Y) \right) = A(X) \otimes_K A(Y)$$

This proves the second claim contingent on the first, but also we know that $A(X), A(Y)$ are reduced since $I(X), I(Y)$ are radical. Then the fact that $A(X) \otimes_K A(Y)$ is reduced follows from the general fact that $U \otimes_K V$ is reduced whenever U, V are reduced K-algebras for K a perfect field. □

1.3 The Zariski Topology

We previously saw that arbitrary intersections and finite unions of affine varieties were again varieties. This is the same property as the closed subsets of a topological space, which leads us to define the Zariski topology on $\mathbb{A}^n$. One immediate goal with this definition will be to decompose varieties into their irreducible components.

Definition 1.9

The **Zariski topology** on $\mathbb{A}^n$ is the topology whose closed sets are the affine varieties in $\mathbb{A}^n$. The Zariski topology on a variety $X \subseteq \mathbb{A}^n$ is the subspace topology of $\mathbb{A}^n$, whose closed sets are the subvarieties of X.

Example 1.4

The Zariski topology on $\mathbb{A}^1$ is the cofinite topology.

The Zariski topology is fairly pathological as a topological space.

Example 1.5

Let $a_1, a_2, \dots$ be a non-repeating sequence of points in $\mathbb{A}^1$. Fix $a \in \mathbb{A}^1$ and an open neighborhood U containing a. Then the complement of U is finite, so eventually every point in the sequence lies in U. Therefore (a_i) converges to every point of $\mathbb{A}^1$.

Example 1.6

Let $f : \mathbb{A}^1 \to \mathbb{A}^1$ be injective. Then any closed set is finite, and thus has a finite preimage, which is closed. So f is continuous.

Example 1.7

The Zariski topology on $\mathbb{A}^2$ is not the product topology on $\mathbb{A}^1 \times \mathbb{A}^1$. To see this, let

$$\Delta = \{(a, a) : a \in K\} \subseteq \mathbb{A}^2$$

Δ is clearly closed in the Zariski topology on $\mathbb{A}^2$. If Δ is closed in the product topology, then the complement can be written as the union of rectangular sets:

$$\mathbb{A}^2 \setminus \Delta = \bigcup_i (U_i \times V_i), \quad U_i, V_i \in \mathcal{O}(\mathbb{A}^1)$$

But for any U, V open, U, V are cofinite and $\mathbb{A}^1$ is infinite, so there is $x \in U \cap V$. This means that $(x, x) \in U \times V$, contradicting the decomposition of $\mathbb{A}^2 \setminus \Delta$.

Proposition 1.19

A space X is Hausdorff if and only if the diagonal is closed in the product topology on $X \times X$.

Proof. We want to show that $\Delta^c = (X \times X) \setminus \Delta$ is open. Let $(x, y) \in \Delta^c$. Since X is Hausdorff there are $U \ni x, V \ni y$ disjoint, so that $U \times V$ contains (x, v) and does not intersect the diagonal. Unioning over all pairs in Δ^c, Δ^c is open so Δ is closed. The same logic works in the other direction. □

This gives a faster proof that the Zariski topology is not the product topology. Also, we will later see that this extends to higher products $\mathbb{A}^{n+m}$, since $U \cap V$ will always be nonempty for appropriate U, V, even outside the cofinite topology.

Definition 1.10

Let X be a topological space. X is **reducible** if it can be written as the union $X = X_1 \cup X_2$ for two proper closed subsets. It is **disconnected** if it can be written as the union of two nonempty disjoint closed subsets.

If X is disconnected then it is reducible, so if X is irreducible then it is connected.

Proposition 1.20

Let X be irreducible. Then

(a) Any two nonempty open sets have nonempty intersection.

(b) Any nonempty open set is dense in X.

Proof. (a) If U_1, U_2 are nonempty and open with empty intersection, then $X \setminus U_1, X \setminus U_2$ are proper closed subsets whose union is all of X.

(b) A set is dense if and only if it intersects every nonempty open set, so this follows from (a).

□

We want ways to describe the irreducible and connected ideals cleanly. Topologically this is difficult, so we will convert their definitions to algebraic ones.

Proposition 1.21

A nonempty affine variety $X \subseteq \mathbb{A}^n$ is irreducible if and only if $A(X)$ is a domain (which is equivalent to $I(X)$ being a prime ideal).

Proof. ($\implies$) Let X be irreducible, take $f, g \in A(X)$ with $fg = 0$, and let $X_1 = V_X(\langle f \rangle), X_2 = V_X(\langle g \rangle)$. Then $X = X_1 \cup X_2$ which are both closed, so $X_1 = X$ or $X_2 = X$. In the first case $f = 0$ and in the second $g = 0$.

($\impliedby$) Let $X = X_1 \cup X_2$ with X_1, X_2 closed. Then X_1, X_2 are both subvarieties of X, so let $I_1 = I_X(X_1), I_2 = I_X(X_2)$. Then $I_1 I_2$ is the zero ideal in $A(X)$, which implies that at least one is zero. But if I_1 is zero then $X_1 = X$ and similarly for I_2. □

This tells us that the Nullstellensatz restricts to a bijective correspondence between nonempty irreducible affine subvarieties of $Y \subseteq \mathbb{A}^n$ to prime ideals of $A(Y)$.

Example 1.8

A finite set is irreducible if and only if it is a single point. In this case $A(X) = K$ which is a domain.

Proposition 1.22

$\mathbb{A}^n$ is irreducble (hence connected).

Proof. $A(\mathbb{A}^n) = K[x_1, \ldots, x_n]$ is an integral domain. □

1.4 Irreducible Decompositions

To show that we can decompose any affine variety into irreducible components, we will prove this for more general topological spaces. This will allow us to easily extend to other forms of varieties later.

Definition 1.11

A topological space X is **Noetherian** if there is no infinite strictly decreasing chain of closed subsets

$$X_0 \supsetneq X_1 \supsetneq X_2 \supsetneq \ldots$$

Recall that Noetherian rings satisfy the *ascending* chain condition on ideals; here we have a *descending* chain condition since we work over closed (equivalently, a Noetherian space satisfies the ascending condition on opens).

Proposition 1.23

Quotients of Noetherian rings are Noetherian.

Proof. Ideals in the quotient correspond to ideals in the original ring (containing the quotiented ideal), so an ascending chain in the quotient lifts to an ascending chain in the original. □

Lemma 1.24

An affine variety $X \subseteq \mathbb{A}^n$ is Noetherian.

Proof. Suppose there is a descending chain

$$X_0 \supsetneq X_1 \supsetneq X_2 \supsetneq \dots$$

Then by the inlusion reversing bijection, we get

$$I(X_0) \subsetneq I(X_1) \subsetneq \dots \subseteq A(X)$$

But $A(X)$ is a quotient of $K[x_1, \dots, x_n]$, which is Noetherian, hence $A(X)$ is Noetherian. This is a contradiction. □

Proposition 1.25

Any subset of a Noetherian space is Noetherian under the subspace topology.

Proof. Let $X \subseteq Y$ with Y Noetherian. Suppose there is some descending chain

$$X_0 \supsetneq X_1 \supsetneq X_2 \supsetneq \dots$$

of relatively closed sets in X. Then by definition there are $Y_i \subseteq Y$ closed such that $Y_i \cap X = X_i$. It follows that

$$Y_0 \supseteq (Y_0 \cap Y_1) \supseteq \dots$$

is a descending chain (not necessarily strict) in Y. To see that this is closed, we have

$$\left(\bigcap_{i=0}^{n} Y_i\right) \cap X = \bigcap_{i=0}^{n}(Y_i \cap X) = \bigcap_{i=0}^{n} X_i = X_n$$

so we actually have

$$Y_0 \supsetneq (Y_0 \cap Y_1) \supsetneq \dots$$

This is a strict descending chain of closed sets in Y, contradicting the assumption that it is Noetherian. □

Proposition 1.26

Any Noetherian topological space X can be written as the finite union of nonempty irreducible closed subsets

$$X = \bigcup_{i=1}^{r} X_i$$

If we also assume that $X_i \nsubseteq X_j$ for $i \neq j$, then this is unique up to permutation.

Proof. If X is empty then we take $r = 0$ and are done. Otherwise, assume that X is not a finite union of irreducibles. Then X itself is certainly not irreducible, so there are proper nonempty closed subsets X_1, X_2 with $X_1 \cup X_2 = X$. These cannot both be irreducible, so one (say X_1) can also be written as the union of two proper nonempty closed subsets. This gives a strictly descending chain of closed subsets, contradicting Noetherianity. So X must have been a finite union of irreducibles.

To see uniqueness, take two decompositions

$$X = \bigcup_{i=1}^{r} X_i = \bigcup_{j=1}^{r} Y_j$$

Then for any X_i we can intersect against the other decomposition and apply irreducibility:

$$X_i = \bigcup_{j=1}^{r} (Y_j \cap X_i)$$

By irreducbility there is j such that $Y_j \cap X_i = X_i$, or $X_i \subseteq Y_j$. Then applying the decomposition in the other direction we get i' s.t. $Y_j \subseteq X_{i'}$. But $X_i \subseteq X_{i'}$ is only satisfied by $i = i'$. Thus we have $X_i = Y_j$, giving a bijection between the decompositions, so they are unique up to permutation. □

Now that we have proven that X can be decomposed into irreducible components in the topological sense, we return to affine varieties and the question of how to actually compute this decomposition.

Definition 1.12

A **primary ideal** I of a ring R is an ideal such that whenever $xy \in I$, either x or a power of y is in I.

Equivalently, $xy \in I$ if and only if either x or y is in I, or if both x, y are in $\sqrt{I}$.

Proposition 1.27

Radicals of primary ideals are prime.

Proof. Let $xy \in \sqrt{I}$ for I primary. Then $(xy)^m \in I \implies x^m y^m \in I$. We either have $x^m \in I$ or $y^m \in I$, or else both x^m, y^m are in $\sqrt{I}$. In every case at least one of x, y is in $\sqrt{I}$. □

In Section A.2, we show that in a Noetherian ring, every ideal (in particular, $I(X)$) has a *primary decomposition*

$$I(X) = \bigcap_{i=1}^{r} Q_i$$

of primary ideals Q_i. Then this is

$$I(X) = \sqrt{I(X)} = \sqrt{\bigcap_{i=1}^{r} Q_i} = \bigcap_{i=1}^{r} \sqrt{Q_i}$$

since if y lies in the radical of the intersection, then y^{m_i} lies in each Q_i for some m_i, hence $y \in \sqrt{Q_i}$ for each i.

Define $P_i = \sqrt{Q_i}$. These ideals are prime since the radicals of prime ideals are prime. So

$$I(X) = \bigcap_{i=1}^{r} P_i \implies X = V(I(X)) = \bigcup_{i=1}^{r} V(P_i)$$

Each $V(P_i)$ is irreducible since P_i is prime. Discard any P_i which are contained in another P_j (i.e. those maximal w.r.t. inclusion within the decomposition). Then this is a decomposition into irreducible closed subsets, which do not contain each other, and is therefore the unique decomposition.

Proposition 1.28

If X is an affine variety, then the irreducible components X_i of X are the maximal (w.r.t. inclusion) irreducible subvarieties of X.

Proof. Suppose not. Then writing

$$X = \bigcup_{i=1}^{r} X_i$$

there must be some X_j which is not maximal. Suppose without loss of generality that this is X_1. Then let $X_1 \subsetneq X_1'$ where X_1' is also an irreducible subvariety. Then

$$X = X_1' \cup \bigcup_{i=2}^{r} X_i$$

is another decomposition into irreducibles, and after possibly discarding any subvarieties contained in X_1', we get a distinct decomposition, contradiction. □

Corollary 1.29

There is a bijection between irreducible components of X and minimal prime ideals of $A(X)$.

Note that starting from the irreducible decomposition of a Noetherian space, we can obtain a decomposition into its connected components. We do this by making a graph where each vertex is an irreducible component, and adding edges between components which intersect. Then the path connected components of the graph are the connected components of X.

As before, we would like to translate this to an algebraic condition.

Theorem 1.30: Chinese Remainder Theorem

If I_1, I_2 are ideals in R that are coprime ($I_1 + I_2 = R$), then the natural ring homomorphism

$$R/_{I_1 \cap I_2} \to \left(R/_{I_1}\right) \times \left(R/_{I_2}\right)$$

is an isomorphism.

Proposition 1.31

Suppose X is a disconnected affine variety with disjoint closed subsets $X_1 \cup X_2 = X$. Then the map $A(X) \to A(X_1) \times A(X_2)$

$$f \mapsto f|_{X_1} \times f|_{X_2}$$

is an isomorphism.

Proof. We know that

$$A(X_i) = A(X)/_{I_X(X_i)}$$

The map given is the natural ring homomorphism

$$A(X) \to \left(A(X)/_{I(X_1)}\right) \times \left(A(X)/_{I(X_2)}\right)$$

Since $X = X_1 \cup X_2$, we have $0 = I(X) = I(X_1) \cap I(X_2)$. Since $\varnothing = X_1 \cap X_2$ we have $A(X) = I(\varnothing) = \sqrt{I(X_1) + I(X_2)}$ so $I(X_1) + I(X_2) = A(X)$. Then we apply the Chinese remainder theorem to see that

$$A(X)/_{I(X_1) \cap I(X_2)} = A(X) \to \left(A(X)/_{I(X_1)}\right) \times \left(A(X)/_{I(X_2)}\right)$$

is an isomorphism. □

So we see that functions on disconnected varieties are just arbitrary combinations of functions on their connected components, so we can essentially just study the rings of functions on connected components.

1.5 Dimensions of Algebraic Varieties

So far we have not formally defined dimension for affine varieties, although examples like curves and surfaces may give us some intuition. Here, as with our previous work, we define this notion topologically and then calculate it using algebra.

Definition 1.13

Let X be a nonempty topological space. Then the **dimension** of X is the supremum over all n such that there is a chain of length n of irreducible closed subsets:

$$\varnothing \subsetneq Y_0 \subsetneq Y_1 \subsetneq \ldots \subsetneq Y_n \subseteq X$$

If $Y \subseteq X$ is a closed irreducible subset then its **codimension** is the supremum over n such that there is a chain of length n of irreducible closed subsets containing Y:

$$Y \subseteq Y_0 \subsetneq Y_1 \subsetneq \ldots \subsetneq Y_n \subseteq X$$

Example 1.9

The only irreducible closed subsets of $\mathbb{A}^1$ are points and $\mathbb{A}^1$, so it has dimension 1.

Note that a general Noetherian topological space is not necessarily of finite dimension:

Example 1.10

Consider $\mathbb{N}$ where the closed sets are of the form $\{1, \ldots, n\}$. This is Noetherian but it has an infinite chain of irreducible closed subsets.

However, for affine varieties, the ring properties will force us to have finite dimension:

Definition 1.14

Let R be a ring. Its **Krull dimension** is the supremum over all n such that there is a chain of length n of prime ideals

$$R \supseteq P_0 \supsetneq P_1 \supsetneq \ldots \supsetneq P_n$$

The **height** of a prime ideal P is the supremum over all n such that there is a chain of length n of prime ideals containing P:

$$R \supseteq P_0 \supsetneq P_1 \supsetneq \ldots \supsetneq P_{n-1} \supseteq P_n$$

Lemma 1.32

If X is a nonempty irreducible subvariety of an affine variety X, then

(a) $\dim X$ is the Krull dimension of $A(X)$,

(b) $\operatorname{codim}_X Y$ is the height of $I_X(Y)$ in $A(X)$.

(c) Both are finite numbers.

Proof. Any chain in $A(X)$ bijects with a chain in X. The lengths are bounded because

$A(X)$ is finitely generated as a K-algebra. □

Lemma 1.33

If R, S are finitely generated K-algebras which are also domains, then

$$\dim(R \otimes_K S) = \dim R + \dim S$$

We just give a sketch; the point is that we can produce a **Noether normalization** which is an injection from

$$K[x_1, \ldots, x_n] \to R$$

which makes R a finitely generated module over the image (where $n = \dim R$). We do this for both R, S, giving a map

$$K[x_1, \ldots, x_n, y_1, \ldots, y_m] \to R \otimes_K S$$

which is a Noether normalization.

Proposition 1.34

Let X, Y be nonemptpy irreducible affine varieties. Then:

(a) $\dim(X \times Y) = \dim X + \dim Y$,

(b) If $Y \subseteq X$ then $\dim X = \dim Y + \operatorname{codim}_X Y$,

(c) If $f \in A(X)$ is nonzero, then every irreducible component of $V_X(f)$ has codimension 1.

Proof. (a) We know that $A(X \times Y) = A(X) \otimes_K A(Y)$. Then the result follows by the Lemma.

(b) $A(X)$ is nice enough that all of the maximal chains of prime ideals have the same length. So then since Y is irreducible, $I_X(Y)$ is prime and the chains in X with $I_X(Y)$ as a step can be decomposed into a length $\operatorname{codim}_X Y$ segment and a length $\dim Y$ segment.

(c) Krull's principal ideal theorem says that every minimal prime ideal containing a (nonunit, non-zero divisor) f has height 1. □

In particular, if X is irreducible then $\operatorname{codim}_X \{a\} = \dim X$ for any point $a \in X$. Now, we develop a similar relation for reducible varieties.

Proposition 1.35

Let X be a topological space and $A \subseteq X$. Then $\dim A \leq \dim X$.

Proof. Any chain of closed irreducibles in A can be written as

$$\varnothing \subsetneq (A \cap X_0) \subsetneq \ldots \subsetneq (A \cap X_n) \subseteq A$$

where the X_i are closed. The X_i can also be taken to be irreducible, and then we have that

$$\varnothing \subsetneq \overline{A \cap X_0} \subsetneq \ldots \subsetneq \overline{A \cap X_n} \subseteq A$$

The inclusions are strict because we can take points in $X_n \setminus X_{n-1}$. □

Proposition 1.36

Let X be a Noetherian space and Y an irreducible subset of X. Then one of the irreducible components of X contains Y.

Proof. Suppose the decomposition is

$$X = X_1 \cup \ldots \cup X_n$$

Then

$$Y = \overline{Y \cap X_1} \cup \ldots \overline{Y \cap X_n}$$

Since Y is reducible, there is some i such that $\overline{Y \cap X_i} = Y$. But $\overline{Y \cap X_i} = Y \cap \overline{X_i} = Y \cap X_i$ so we have $Y \subseteq X_i$. □

Proposition 1.37

(a) If X has irreducible decomposition $X_1 \cup \ldots \cup X_n$, and X is a Noetherian topological space, then

$$\dim X = \max \{\dim X_i\}$$

(b) For any topological space,

$$\dim X = \sup_{a \in X} \{\operatorname{codim}_X \{a\}\}$$

Proof. (a) We automatically have $\dim X \geq \max \{\dim X_i\}$. Now, take a chain of length $r = \dim X$:

$$\varnothing \subsetneq Y_0 \subsetneq \ldots \subsetneq Y_r \subseteq X$$

Then Y_r is contained in some X_i, so we get $r \geq \dim X_i$.

(b) This is clear from the formula. □

Definition 1.15

A Noetherian topological space is of **pure dimension** if every irred. component has the same dimension. An affine variety of dimension 1 is a **curve**, of dimension 2 a **surface**, and of codimension 1 a **hypersurface**.

Chapter 2

Sheaves and Morphisms

2.1 Regular Functions on Affine Varieties

As when developing a theory of categories like topological spaces, differentiable manifolds, and so on, we would like to describe what the morphisms in the category of affine varieties should be. We will begin with functions $f : X \to \mathbb{A}^1$, since these are the mostly closely related to polynomial functions, hence easiest to describe in language related to what we have already seen. Taking continuous and differentiable maps as our examples, we note that the property of being a continuous or differentiable map is something that can be checked locally, and is preserved under restrictions to smaller domains. Moreover, if two such maps agree on their open intersection, then they "glue" into a map on the union of their domains. We will define maps from $X \to \mathbb{A}^1$ in such a way that these properties also hold.

Perhaps the most immediately obvious maps to consider are the polynomial functions. However, if we work only with maps defined on open sets, then we have access to a richer class of maps. Recall that by definition, open sets are the complement of the vanishing locus of some functions. As a result, if U is open in X and $f \in I(X \setminus U)$, then for $g \in A(X)$, the expression

$$\varphi = \frac{g}{f}$$

has nonzero denominator in U, so at least this expression makes sense to write down. In order to mimic the local and gluing properties of continuous and differentiable maps, though, we will only require that the functions are *locally* given in the form of g/f, as opposed to globally. This isn't necessarily justification for why these are the right maps to look at, but it turns out that they are.

Definition 2.1

Let X be an affine variety and $U \subseteq X$ open. Then a **regular function** on U is a function $\varphi : U \to \mathbb{A}^1$ such that for every $a \in U$ there is $f, g \in A(X)$ and an open subset $a \in U_a \subseteq U$ such that

$$\varphi(x) = \frac{g(x)}{f(x)}$$

on U, with $f(x) \neq 0$ on U. We denote the set of regular functions on U by $\mathcal{O}_X(U)$. This carries the structure of a K-algebra.

Example 2.1

Let $X = V(x_1x_4 - x_2x_3)$ and let $U = X \setminus V(x_2, x_4)$. Then define

$$\varphi(x_1, x_2, x_3, x_4) = \begin{cases} \frac{x_1}{x_2}, & x_2 \neq 0 \\ \frac{x_3}{x_4}, & x_4 \neq 0 \end{cases}$$

When x_2, x_4 both nonzero then $x_1x_4 - x_2x_3$ implies that this is well defined. But neither quotient makes sense on all of $V(x_1x_4 - x_2x_3)$ (and more generally, there is no global representation).

Lemma 2.1

Let $U \subseteq X$ be open and let $\varphi \in \mathcal{O}_X(U)$. Then

$$V(\varphi) = \{x \in U : \varphi(x) = 0\}$$

is closed in U.

Proof. We recall that if $\{U_i : i \in I\}$ is an open cover of U, then $V \subseteq U$ is closed in U if and only if $V \cap U_i$ is closed in U_i for each i. Now, for each $a \in U$ we have $U_a \subseteq U$ open and f, g with $\varphi = g/f$ on U_a. Since f is nonvanishing, we have

$$V(\varphi) \cap U_a = V(g) \cap U_a$$

which is closed in U_a. Since the collection of all such U_a covers U, $V(\varphi)$ is closed in U. □

As desired, we have maintained the restriction property for regular functions. Precisely, if $U \subseteq V \subseteq V$ and U, V are open, then the restriction map $\mathcal{O}_X(V) \to \mathcal{O}_X(U)$ takes regular functions to regular functions. However, this is not surjective in general. For instance, the element $\varphi(x) = 1/x$ is a regular function in $\mathcal{O}_{\mathbb{A}^1}(\mathbb{A}^1 \setminus \{0\})$ but there is no regular function which restricts to φ from $\mathcal{O}_{\mathbb{A}^1}(\mathbb{A}^1)$. We will later see a theorem that implies this occurs for a broad class of removed singularities. Nevertheless, the restriction is actually injective under the right assumptions.

Proposition 2.2: Identity Theorem

Let X be an irreducible affine variety and let $\varnothing \subsetneq U \subseteq V \subseteq X$ be open. Then if $\varphi_1, \varphi_2 \in \mathcal{O}_X(V)$ and $\varphi_1|_U = \varphi_2|_U$, it follows that $\varphi_1 = \varphi_2$

Proof. The set $V(\varphi_1 - \varphi_2)$ on which φ_1, φ_2 is closed. It contains U, which is nonempty and open in V. V is irreducible since it is an open subset of X irreducible. So the closure of U in V is V, hence $V \subseteq V(\varphi_1 - \varphi_2)$. Thus $\varphi_1 = \varphi_2$ on V. $\square$

This resembles analytic continuation from complex analysis in spirit. However, it is a less surprising result because we already know that open sets are "large" in the Zariski topology, so it is natural that they extend uniquely.

Now, we compute $\mathcal{O}_X(U)$ for a class of simple open sets.

Definition 2.2

Let X be an affine variety and $f \in A(X)$. Then the **distinguished open subset** for f in X is

$$D(f) = \{x \in X : f(x) \neq 0\}$$

A general open subset is called a distinguished open subset if it is of the form $D(f)$.

Proposition 2.3

The intersection of distinguished open subsets is a distinguished open subset.

Proof. Let $f, g \in A(X)$. Then

$$D(f) \cap D(g) = (X \setminus V_X(f)) \cap (X \setminus V_X(g)) = X \setminus (V_X(f) \cup V_X(g)) = X \setminus V(fg) = D(fg) \quad \square$$

Proposition 2.4

Any open subset in X is a finite union of distinguished subsets.

Proof. Let $U \subseteq X$ be open. Then $U = X \setminus V(I)$ for some ideal $I \subseteq A(X)$, and by Hilbert's basis theorem we know this can be written as

$$U = X \setminus V_X(\langle f_1, \ldots, f_r \rangle)$$

for some finite generators. Then

$$U = X \setminus \bigcap_{i=1}^{r} V_X(f_i) = \bigcup_{i=1}^{r} (X \setminus V_X(f_i)) = \bigcup_{i=1}^{r} D(f_i) \quad \square$$

Thus the distinguished open sets form a basis for the Zariski topology. This is helpful because we showed earlier that being a regular function is a local property. Hence, we can write an open set as a union of distinguished opens, and check if the restriction to each is regular. Thus, we just need to understand $\mathcal{O}_X(D(f))$.

Lemma 2.5: Partition of Unity

Let X be an affine variety and suppose

$$D(f) = \bigcup_{i \in I} D(f_i)$$

Then there is $m \in \mathbb{N}$ such that

$$f^m = \sum_{i \in I_0} r_i f_i$$

where I_0 is a finite subset of I and each $r_i \in A(X)$.

This is basically the Nullstellensatz, so we remark that algebraic closure of the ground field is critical for the existence of partitions of unity. Since each f_i is supported on $D(f_i)$, this breaks f^m into terms which are supported on distinguished open sets. The utility of this is clear from analogy with smooth partitions of unity.

Proof. By the Nullstellensatz,

$$f \in I_X(V_X(f))$$

We want to show that this is the radical of an ideal which is the sum of some principal ideals. Note that

$$V_X(f) = X \setminus D(f) = \bigcap_{i \in I} V_X(f_i) = V_X(\langle f_i : i \in I \rangle)$$

So

$$f \in \sqrt{\langle f_i : i \in I \rangle} \implies f^m = \sum_{i \in I_0} r_i f_i$$

for some finite subset I_0. ☐

Functions on distinguished open sets are locally quotients of polynomials except on $V_X(\langle f \rangle)$, which we know has dimension at most 1 by Krull's principal ideal theorem. In some sense, we should expect that this will allow the singularities to act similarly to removable singularities.

Proposition 2.6

For any distinguished open subset $D(f) \subseteq X$, the regular functions on $D(f)$ take the form

$$\mathcal{O}_X(D(f)) = \left\{ \frac{g}{f^m} : g \in A(X), m \in \mathbb{N} \right\}$$

Thus, we see that regular functions on distinguished open subsets are given globally as a polynomial quotient, not just locally. In particular, taking $f = 1$ we get $D(f) = X$ so the regular functions on X are all of $A(X)$.

Proof. ($\supseteq$) This is clear, since any quotient g/f^m makes sense on $D(f)$ and is clearly locally a quotient.

($\subseteq$) Let $\varphi \in \mathcal{O}_X(D(f))$.

Claim 1: for any $a \in D(f)$, φ can be written locally as a quotient

$$\varphi = \frac{g_a}{f_a}$$

on U_a where $D(f_a) = U_a$ and $g_a = 0$ on $V_X(f_a)$.

To see this, first note that we can generally write

$$\varphi = \frac{g_a}{f_a}$$

on U_a for some g_a, f_a with $a \in U_a$. But since the distinguished opens form a basis, there is a distinguished open subset $D(h_a)$ contained in U_a which contains a. On $D(h_a)$,

$$\varphi = \frac{g_a h_a}{f_a h_a}$$

Since $D(h_a) \subseteq D(f_a)$, $V_X(h_a) \supseteq V_X(f_a)$, so h_a vanishes wherever f_a vanishes. So on $V_X(f_a h_a)$, $g_a h_a = 0$. Thus taking $g'_a = g_a h_a$, $f'_a = f_a h_a$, we have that $g'_a = 0$ on $V_X(f'_a)$, proving Claim 1.

Claim 2: For $a, b \in D(f)$ and f_a, g_a, f_b, g_b as constructed above, $g_a f_b = g_b f_a$ as elements of $A(X)$.

To see this, $U_a \cap U_b$ is a nonempty open set, on which φ has two representations

$$\varphi = \frac{g_a}{f_a} = \frac{g_b}{f_b}$$

Clearing denominators, we have $g_a f_b = g_b f_a$ on $U_a \cap U_b$. On the other hand, if $x \notin U_a \cap U_b$, then either $x \in V_X(f_a)$ or $x \in V_X(f_b)$ (since by Claim 1, we chose $U_i = X \setminus V_X(f_i)$). Say $x \in V_X(f_a)$. Then by Claim 1, $g_a(x) = f_a(x) = 0$, so both sides are zero. So everywhere we have $g_a f_b = g_b f_a$, proving Claim 2.

Finally, with g_a, f_a, U_a defined as in Claim 1,

$$\bigcup_{a \in D(f)} U_a = \bigcup_{a \in D(f)} = \bigcup_{a \in D(f)} D(f_a)$$

So by the partition of unity, there is m and a finite subset $A \subseteq D(f)$ such that

$$f^m = \sum_{a \in A} r_a f_a$$

Now, define

$$g = \sum_{a \in A} r_a g_a$$

It follows that for any $x \in D(f)$,

$$g f_x = \sum_{a \in A} r_a g_a f_x \overset{(2)}{=} \sum_{a \in A} r_a g_x f_a = g_x \sum_{a \in A} r_a f_a = g_x f^m$$

Since f, f_x are nonzero at x, we have

$$\frac{g(x)}{f^m(x)} = \frac{g_x}{f_x} = \varphi(x)$$

□

We can view this construction in a completely algebraic sense.

Definition 2.3

Let $S \subseteq R$ be a multiplicative subset. Then the **localization** of R at S is the ring of fractions

$$S^{-1}R = R[S^{-1}] = \left\{ \frac{r}{s} : r \in R, s \in S \right\} \Big/ \sim$$

where the equivalence relation $\sim$ is such that

$$\frac{r}{s} \sim \frac{r'}{s'}$$

if and only if there is $t \in S$ such that

$$t(rs' - r's) = 0$$

In particular, when S is the subset $S = \{1, f, f^2, \ldots\}$ for some $f \in R$, we denote $R_f = R[S^{-1}]$.

Corollary 2.7

Let $f \in A(X)$. Then the ring of regular functions on the distinguished subset $D(f)$ is

$$\mathcal{O}_X(D(f)) \cong A(X)_f$$

which are isomorphic as K-algebras.

Proof. We can consider the homomorphism of K-algebras $\varphi : A(X)_f \mapsto \mathcal{O}_X(D(f))$ given by

$$\varphi\left(\left[\frac{g}{f^m}\right]\right) = \frac{g}{f^m}$$

We check that this is well defined, injective, and surjective. First, if

$$\left[\frac{g}{f^m}\right] = \left[\frac{g'}{f'^n}\right]$$

then there is k such that

$$f^k \left(g f'^n - g' f^m\right) = 0$$

Since $f \neq 0$ on $D(f)$, we have $gf'^n - g'f^m = 0$ on $D(f)$, so

$$\frac{g}{f^m} = \frac{g'}{f'^n}$$

as functions on $D(f)$. So this map is well defined. Since $D(f)$ os a distinguished subset, every element of $\mathcal{O}_X(D(f))$ is of the form g/f^m, so this map is surjective. To prove injectivity, if

$$\varphi\left(\left[\frac{g}{f^m}\right]\right) \equiv 0$$

on $D(f)$ then $g = 0$ on $D(f)$, so $fg = 0$ on $D(f)$. Then

$$\left[\frac{g}{f^m}\right] = \left[\frac{0}{1}\right]$$

since $f(g \cdot 1 - f^m \cdot 0) = fg = 0$ on $D(f)$. □

So we have computed the ring of regular functions on distinguished open subsets. As an example of a subset which is not distinguished, we consider $\mathbb{A}^2 \setminus \{(0,0)\}$.

Example 2.2

We will show that

$$\mathcal{O}_{\mathbb{A}^2}(\mathbb{A}^2 \setminus \{(0,0)\}) \cong \mathbb{K}[x_1, x_2] \cong \mathcal{O}_{\mathbb{A}^2}(\mathbb{A}^2)$$

which is to say that every regular function on $U = \mathbb{A}^2 \setminus \{(0,0)\}$ can be extended over the origin. To see this, note that $U = D(x_1) \cup D(x_2) \supseteq D(x_1x_2)$. Letting $\varphi \in \mathcal{O}_{\mathbb{A}^2}(U)$, φ restricts to regular functions on $D(x_i)$, which then restrict to regular functions on $D(x_1x_2)$. This means there are m, n such that

$$\varphi = \frac{g_1}{x_1^m} = \frac{g_2}{x_2^n}$$

for $g_1, g_2 \in K[x_1, x_2]$. The equality holds on $D(x_1x_2)$, so by our characterization of $\mathcal{O}_X(D(x_1x_2))$ we know the two are equal as equivalence classes. Thus there is r such that

$$(x_1x_2)^r(g_1x_2^n - g_2x_1^m) = 0$$

as elements of $K[x_1, x_2]$. Since $K[x_1, x_2]$ is a domain, we have

$$g_1x_2^n = g_2x_1^m$$

Since $K[x_1, x_2]$ is a UFD, we get that x_1^m divides x_2^n or g_1. By assuming $\gcd(g_1, x_1) = \gcd(g_2, x_2) = 1$ if necessary, which is harmless, we see that $m = 0, n = 0$, so $\varphi = g_1 = g_2$ extends to a polynomial function on $\mathbb{A}^2$.

Proposition 2.8

Let Y be a nonempty irreducible subset of an affine variety X such that $A(X)$ is a UFD. Let $U = X \setminus Y$. Then $\mathcal{O}_X(U) = A(X)$ if and only if $\operatorname{codim}_X(Y) \geq 2$.

Proof. □

2.2 Sheaves

We defined regular functions in such a way that:

1. Restrictions of regular functions are regular,
2. Regularity can be checked locally, meaning that if a function is regular on each open set in an open cover, then it is a regular function everywhere,
3. Constructing regular functions by "gluing" together functions on open subsets is unique (by the identity theorem).

The notion of a sheaf is an abstract view of how data of this type can be collected into a single object. We will formalize this so that we can discuss mappings from an affine variety into another one (other than $\mathbb{A}^1$, which we have just developed), seeing affine varieties as not simply a set in $\mathbb{A}^n$ but as such a set together with the data given by the sheaf.

Definition 2.4

Let (X, τ) be a topological space. Then a **presheaf** of rings on X is a collection of rings $\{\mathcal{F}(U)\}_{U \in \tau}$, together with ring homomorphisms $\rho_{V \to U} : \mathcal{F}(V) \to \mathcal{F}(U)$ for each inclusion of opens $U \subseteq V$. We also require that $\rho_{V \to V}$ is the identity on $\mathcal{F}(V)$, and that whenever $U \subseteq V \subseteq W$, the following diagram commutes:

$$\mathcal{F}(U) \xrightarrow{\rho_{U \to V}} \mathcal{F}(V) \xrightarrow{\rho_{V \to W}} \mathcal{F}(W), \qquad \mathcal{F}(U) \xrightarrow{\rho_{U \to W}} \mathcal{F}(W)$$

Elements of $\mathcal{F}(V)$ are called **sections** of $\mathcal{F}$ on V, and elements of $\mathcal{F}(X)$ are called **global sections**. The maps $\rho_{V \to U}$ are also called **restriction maps**, and $\rho_{V \to U}(\varphi)$ is denoted $\varphi|_U$.

$\mathcal{F}(U)$ should intuitively be thought of as a ring of functions with the restriction maps as actual restrictions, but we do not require this.

Definition 2.5

A presheaf $\mathcal{F}$ is called a **sheaf** if it satisfies the gluing properties:

1. Whenever $\{U_i\}_{i \in I}$ is an open cover of $U \subseteq X$ open, if $\varphi_i \in \mathcal{F}(U_i)$ for each i and $\varphi_i|_{U_i \cap U_j} = \varphi_j|_{U_i \cap U_j} \in \mathcal{F}(U_i \cap U_J)$ for each i, j, then there is $\varphi \in \mathcal{F}(U)$ such that $\varphi|_{U_i} = \varphi_i$,

2. Such a φ is unique.

In essence, compatible data can glued together on small pieces to produce larger data.

We also note that this makes sense in many categories (basically any reasonable category which is a set with structure). For instance, we can have a sheaf of K-algebras, R-modules, abelian groups, and so on.

Example 2.3

The continuous ring-valued functions on a topological space form a sheaf of rings. If they are K-valued, then it is also a sheaf of K-algebras. The differentiable functions on $\mathbb{R}^d$ also form a sheaf of K-algebras.

Example 2.4

If X is any space and R a ring, then the constant functions $f : U \to R$, $U \subseteq X$ form the **constant presheaf**. However, this is in general not a sheaf, because if we can take two disjoint open sets, then we can prescribe different values on each. This is compatible data since the intersection is empty, but it does not glue into a constant function. Intuitively, being constant is not a local property. The *locally* constant functions do form a sheaf, though.

Proposition 2.9

If X is an affine variety, then the sheaf of regular functions $\mathcal{F}(U) = \mathcal{O}_X(U)$ with the restriction maps given by function restriction form a sheaf.

Proof. That this is a presheaf is clear. Since being regular is a local property, compatible regular functions can be glued into regular functions, and by the identity theorem they are unique. □

Definition 2.6

Let $W \subseteq X$ be open and $\mathcal{F}$ a presheaf (resp. sheaf) on X. Then the restriction presheaf (sheaf) on W is

$$\mathcal{F}|_W(U) = \mathcal{F}(U)$$

for $U \subseteq W$ open, with the same restriction maps as in $\mathcal{F}$.

If sheaves of functions collect data of functions, then we can look at the behavior of functions around a given point in the topological space. The possible behaviors of these functions are described by stalks.

Definition 2.7

Let $\mathcal{F}$ be a presheaf over X and $a \in X$. Then the **stalk** of $\mathcal{F}$ at a is

$$\mathcal{F}_a = {\{(U,\varphi) : \varphi \in \mathcal{F}(U)\}}/{\sim}$$

with the equivalence relation

$$(U,\varphi) \sim (V,\psi)$$

if and only if there is $W \subseteq U \cap V$ open containing a such that $\varphi|_W = \psi|_W$. An element of $\mathcal{F}_a$ is called a **germ** at a, or sometimes a **local section**.

A stalk has the structure of a ring if $\mathcal{F}$ is a presheaf of rings, of K-algebras if $\mathcal{F}$ is a presheaf of K-algebras, and so on. For instance, we have

$$[(U_1,\varphi_1)] + [(U_2,\varphi_2)] = [(U_1 \cap U_2, \varphi_1 + \varphi_2)]$$
$$[(U_1,\varphi_1)] \cdot [(U_2,\varphi_2)] = [(U_1 \cap U_2, \varphi_1\varphi_2)]$$

When we work with the sheaf of regular functions, we denote this $\mathcal{O}_{X,a}$. The information contained in a germ or stalk is the local behavior of a function. The strength of this information is dependent on the specific sheaf or presheaf.

Example 2.5

Consider the sheaf of continuous functions on X and consider a germ $\varphi_a \in \mathcal{F}_a$ for $a \in X$. Then every representative of the germ agrees when evaluated at a, so $\varphi_a(a)$ is well defined (though $\varphi_a(a)$ does not uniquely determine a germ). However, a germ cannot be evaluated anywhere else, since there are continuous functions in the same germ with different values there.

Example 2.6

Consider the sheaf of holomorphic functions on $\mathbb{C}$ and consider a germ $\varphi_a \in \mathcal{F}_a$. Then any two representatives agree on an open set containing a. Up to ignoring representatives defined on disconnected sets, regularity tells us that each germ corresponds to exactly one holomorphic function. Thus any section on, say, $\mathbb{D}$, is entirely determined by its germ at 0, and the germ can be evaluated anywhere on its maximal domain.

Example 2.7

For the sheaf of regular functions on an affine variety, there is a well-defined evaluation map $\varphi_a \mapsto \varphi(a)$.

We will now see how to compute the stalks of regular functions over affine varieties. This algebraic operation is called localization, because it reflects the fact that we are looking at the properties of these functions locally.

Definition 2.8

Let R be a ring and $\mathfrak{p} \subseteq R$ an ideal. Let $S_p = R \setminus \mathfrak{p}$ (which is multiplicative). Then the **localization** of R around $\mathfrak{p}$ to be the localization $R_\mathfrak{p} = R[S_\mathfrak{p}^{-1}]$.

Lemma 2.10

Let $\pi : R \to R_\mathfrak{p}$ be the natural inclusion map $r \mapsto \left[\frac{r}{1}\right]$. Then the maps

$$\{\mathfrak{q} \subseteq R_\mathfrak{p} \text{ prime}\} \underset{\mathfrak{q}' \cdot R_\mathfrak{p} \leftarrow\!\shortmid \mathfrak{q}'}{\overset{\mathfrak{q} \mapsto \pi^{-1}(\mathfrak{q})}{\rightleftarrows}} \{\mathfrak{q}' \subseteq R \text{ prime, contained in } \mathfrak{p}\}$$

are inclusion preserving bijections which are inverses.

Definition 2.9

A **local ring** is a ring with a unique maximal ideal.

Corollary 2.11

If $\mathfrak{p}$ is prime then $R_\mathfrak{p}$ is a local ring with maximal ideal $\mathfrak{p} \cdot R_\mathfrak{p}$.

Proof. The unique prime ideal contained in $\mathfrak{p}$ which is maximal among such ideals is $\mathfrak{p}$ itself. Passing through the bijection, the unique maximal ideal in $R_\mathfrak{p}$ is $\mathfrak{p} \cdot R_\mathfrak{p}$. □

Lemma 2.12

Let $a \in X$ be a point in an affine variety. The map from the localization of the coordinate ring to the stalk at a

$$A(X)_{I_X(a)} \to \mathcal{O}_{X,a}$$

given by

$$\left[\frac{g}{f}\right]_{A(X)_{I_X(a)}} \mapsto \left[\left(D(f), \frac{g}{f}\right)\right]_{\mathcal{O}_{X,a}}$$

is an isomorphism of K-algebras.

Proof. We check that this map is well defined, surjective, and injective. Being a K-algebra morphism is clear.

To see that it is well defined, if

$$\left[\frac{g}{f}\right]_{A(X)_{I_X(a)}} = \left[\frac{g'}{f'}\right]_{A(X)_{I_X(a)}}$$

then there is $h \notin I_X(a)$ such that

$$h(fg' - gf') = 0 \in A(X)$$

So on

$$D(h) \cap D(f) \cap D(f')$$

we have

$$\frac{g}{f} = \frac{g'}{f'}$$

pointwise. This is a subset of both $D(f), D(f')$, so

$$\left[\left(D(f), \frac{g}{f}\right)\right]_{\mathcal{O}_{X,a}} = \left[\left(D(f'), \frac{g'}{f'}\right)\right]_{\mathcal{O}_{X,a}}$$

For surjectivity, pick

$$[(U, \varphi)]_{\mathcal{O}_{X,a}}$$

Then φ is regular so there is U_a open containing a and $g, f \in A(X)$ such that $\varphi = g/f$ on U_a and $f \neq 0$ on U_a. In particular we can assume that $U_a = D(f)$. Then we have that

$$\varphi = \frac{g}{f}$$

on $a \in U \cap D(f)$ nonempty. Thus we have that

$$[(U, \varphi)]_{\mathcal{O}_{X,a}} = \left[\left(D(f), \frac{g}{f}\right)\right]_{\mathcal{O}_{X,a}}$$

So this is the image of

$$\left[\frac{g}{f}\right]_{A(X)_{I_X(a)}}$$

To check injectivity, if

$$\left[\left(D(f), \frac{g}{f}\right)\right]_{\mathcal{O}_{X,a}} = \left[\left(X, \frac{0}{1}\right)\right]_{\mathcal{O}_{X,a}}$$

Then there is $U \subseteq D(f)$ containing A such that

$$\left.\frac{g}{f}\right|_U = 0 \in \mathcal{O}_X(U)$$

Since the distinguished opens form a basis for the Zariski topology, there is h such that $D(h) \subseteq U$. In particular, we must have $h \in A(X) \setminus I_X(a)$. Then

$$\left.\frac{g}{f}\right|_{D(h)} = \frac{0}{1} \in \mathcal{O}_X(D(h)) = A(X)_h$$

Therefore by definition, there is m such that

$$h^m(g \cdot 1 - f \cdot 0) = h^m g = 0 \in A(X)$$

Since $h \notin I_X(a)$,

$$\left[\frac{g}{f}\right]_{A(X)_{I_X(a)}} = \left[\frac{0}{1}\right]_{A(X)_{I_X(a)}} \qquad \square$$

Corollary 2.13

$\mathcal{O}_{X,a}$ is a local ring and the unique maximal ideal is given by

$$I_X(a)A(X)_{I_X(a)} = \left\{\frac{g}{f} : f \notin I_X(a), g \in I_X(a)\right\}\Big/\sim \; = \left\{\frac{g}{f} : f(a) \neq 0, f(a) = 0\right\}\Big/\sim$$

where $\sim$ is as for localizations.

Proof. $I_X(a)$ is prime. □

Definition 2.10

The **local ring** of X at a is $\mathcal{O}_{X,a}$ and its maximal ideal is denoted

$$I_a \coloneqq I_X(a)A(X)_{I_X(a)} = \left\{\frac{g}{f} : f(a) \neq 0, f(a) = 0\right\}\Big/\sim$$

2.3 Ringed Spaces and Morphisms

Now that we have a characterization of nice functions on affine varieties, we can see morphisms between them as (continuous) maps which preserve those functions. That is, we want a function to have the property that whenever φ is a regular function on U, then $\varphi \circ f$ is a regular function on $f^{-1}(U)$.

$$X \xrightarrow{\quad f \quad} Y$$

$$\mathcal{O}_X(f^{-1}(U)) \xleftarrow[f^*]{} \mathcal{O}_Y(U)$$

Definition 2.11

A **ringed space** is a topological space with a sheaf of rings $\mathcal{O}_X$ on X. The sheaf is called the **structure sheaf**.

In the case that X is an affine variety we always take the structure sheaf to be the sheaf of regular functions. Any open subset of X is also a ringed space using the restricted sheaf $\mathcal{O}|_U$. For general rings, the composition $\varphi \circ f$ does not necessarily make sense. As a result, we adopt the convention from here until the introduction of schemes that for any ringed space, $\mathcal{F}(U)$ is always a subset of the set of functions from $U \to K$.

Definition 2.12

Let X, Y be ringed spaces and $f : X \to Y$ any map. Then for any $\varphi : U \to K$ for $U \subseteq Y$ open, the map $\varphi \circ f : f^{-1}(U) \to K$ is called the **pullback** of φ by f and is denoted $f^*\varphi$. If f is continuous and for any $U \subseteq Y$ open, $\varphi \in \mathcal{O}_Y(U)$, we have $f^*\varphi \in \mathcal{O}_X(f^{-1}(U))$, then f is called a **morphism** of ringed spaces. f is an isomorphism of ringed spaces if it is bijective and f^{-1} is also a morphism. We say that a map $f : X \to Y$ continuous for X, Y affine varieties is a **morphism** of affine varieties if it is a morphism as a ringed space.

In the case that f is a morphism of ringed spaces, then $f^* : \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$ is a K-algebra morphism for each U.

Proposition 2.14

The identity map on a ringed space X is a morphism, and if $f : X \to Y$, $g : Y \to Z$ are morphisms of ringed spaces, then $g \circ f : X \to Z$ is a morphism as well.

this tells us that the ringed spaces form a category.

Proposition 2.15

If $f : X \to Y$ is a morphism of ringed spaces and $U \subseteq X, V \subseteq Y$ are open with $V \subseteq f(U)$, then $f|_U : U \to V$ is a morphism of ringed spaces.

In particular, the inclusion map $U \to X$ is a restriction of the identity, so a morphism.

Lemma 2.16

Let $f : X \to Y$ be a map of ringed spaces, and let $\{U_i\}_{i\in I}$ be an open cover of X such that all the restrictions $f|_{U_i} : U_i \to Y$ are morphisms of ringed spaces. Then f is a morphism.

Proof. f is continuous since it is continuous on an open cover. Now, let $V \subseteq Y$ be open, and let $\varphi \in \mathcal{O}_Y(V)$. Then for each i,

$$(f^*\varphi)\,|_{U_i\cap f^{-1}(V)} = \left(f|_{U_i\cap f^{-1}(V)}\right)^*\varphi \in \mathcal{O}_X(U_i \cap f^{-1}(V))$$

by assumption. So the sets $\left\{U_i \cap f^{-1}(V)\right\}_{i\in I}$ are an open cover of $f^{-1}(V)$, on which the sections $f^*\varphi\Big|_{U_i\cap f^{-1}(V)}$ agree on intersections. So by the gluing property there is a unique map which agrees with the restrictions, which must be $f^*\varphi$. So $f^*\varphi$ is a section in $\mathcal{O}_X(f^{-1}(V))$, and therefore f is a morphism. □

Appendix A

Results in Commutative Algebra

This appendix is for results in commutative algebra which are not explicitly covered in the lecture notes. As in the main notes, every ring is assumed to be commutative with unity.

A.1 Hilbert's Basis Theorem

This result is taken from Eisenbud's *Commutative Algebra: with a View toward Algebraic Geometry*. One consequence of the basis theorem is that every ideal in a polynomial ring is finitely generated. This is an important basic result which is used in the proof that every variety is the finite union of irreducibles.

Definition A.1

A ring is said to be **Noetherian** if every strictly ascending chain of ideals terminates, or equivalently if every ideal is finitely generated.

Proposition A.1

These two definitions are equivalent.

Proof. For an ideal $I \subseteq R$, take $r_1 \in R, r_2 \in R \backslash \langle r_1 \rangle, r_3 \in R \backslash \langle r_1, r_2 \rangle$, and so on. Then $\langle r_1 \rangle \subseteq \langle r_1, r_2 \rangle \subseteq \dots$ is strictly ascending, so it terminates. It only terminates if $\langle r_1, \dots, r_k \rangle = I$, so I is finitely generated.

Take $I_1 \subseteq I_2 \subseteq \dots$. Then $\bigcup I_i$ is an ideal, and it is finitely generated. These generators are all included in some I_k, so the chain terminates. □

Example A.1

Fields only have themselves and the zero ideal as ideals, so they are Noetherian. By the Chinese Remainder Theorem, $\mathbb{Z}$ is Noetherian, since any ideal is generated by the gcd.

Example A.2

Every ring is finitely generated as an ideal (since $R = \langle 1 \rangle$). However, we can also consider rings which are finitely generated as $\mathbb{Z}$-modules. This means that every element is a finite sum over the generating set S of the form

$$n_1 s_1 + \ldots + n_k s_k, \quad n_i \in \mathbb{Z}, s_i \in S$$

As a result of the basis theorem, we will see that any ring which is finitely generated as a $\mathbb{Z}$-module, then the ring itself is also Noetherian.

Theorem A.2: Hilbert's Basis Theorem

If R is Noetherian then $R[x]$ is as well.

The basic point of the proof is that the leading terms of polynomials effectively act like the ring itself, when they share the same degree. For a polynomial

$$f = a_n x^n + \ldots + a_1 x + a_0$$

a_n is called the leading coefficient and $a_n x^n$ the leading term.

Proof. Pick an ideal I of $R[x]$. Let f_1 be any nonzero polynomial of lowest degree in I. Recursively choose f_i to be a nonzero polynomial of lowest degree in $I \setminus (f_1, \ldots, f_{i-1})$, terminating if this is ever empty. If it does not terminate, then we have infinitely many leading coefficients: denote the leading coefficient of f_j by a_j. Then $(a_1, a_2, \ldots)$ is an ideal in R, and is in particularly finitely generated by $a_1, \ldots, a_m$ for some m. Then we can write

$$a_{m+1} = \sum_{j=1}^{m} a_j u_j$$

for some $u_j \in R$. We can then contradict our lowest degree construction by showing that the leading term of f_{m+1} is actually in $(f_1, \ldots, f_m)$. By increasing the degree of our polynomials, we can cancel the leading term: define

$$g = \sum_{j=1}^{m} u_j a f_j x^{\deg f_{m+1} - \deg f_j}$$

which has leading term equal to $a_{m+1} x^{\deg f_{m+1}}$. Therefore $f_{m+1} - g$ has degree strictly less than f_{m+1}, but is also not $(f_1, \ldots, f_m)$, contradicting our choice of f_{m+1}. □

A.2 Primary Decompositions of Noetherian Rings

This is Atiyah-MacDonald Chapter 7.

Lemma A.3

If R is Noetherian then every ideal is a finite intersection of irreducible ideals.

Proof. If there are some ideals that are not a finite intersection, then the set of such ideals is nonempty and has a maximal element $\mathfrak{a}$. This is reducible so it is the intersection of $\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c}$ which both strictly contain $\mathfrak{a}$. But then $\mathfrak{b}, \mathfrak{c}$ are finite intersections of irreducible ideals and so is $\mathfrak{a}$. □

Lemma A.4

In a Noetherian ring every irreducible ideal is primary.

Proof. We can quotient out by the ring, so this is equivalent to the statement that in any Noetherian ring, the zero ideal is primary if it is irreducible. Suppose the zero ideal is irreducible. Then take x, y with $y \neq 0$ and $xy = 0$. Then we need to show that $x^n = 0$ for some n.

Consider the chain of annihilators

$$\mathrm{Ann}(x) \subseteq \mathrm{Ann}(x^2) \subseteq \dots$$

The ring is Noetherian so there is n such that $\mathrm{Ann}(x^n) = \mathrm{Ann}(x^{n+1})$. We claim that $(x^n) \cap (y) = 0$. Indeed, if $a \in (x^n) \cap (y)$ then $a \in (y)$ so $ax = 0$. If also $a \in (x^n)$ then $a = bx^n$, and $ax = bx^{n+1} = 0$ so $b \in \mathrm{Ann}(x^{n+1}) = \mathrm{Ann}(x^n)$ so $0 = bx^n = a$. Then $(y) \neq 0$ so $(x^n) = 0$, since 0 is irreducble. So 0 is primary. □

The following follows immediately:

Theorem A.5

In a Noetherian ring, every ideal has a primary decomposition.

Definitions